Outline

- Vector product (cross product)
- **Rigid body**
  - **Rigid body and its properties**
  - Freedom of rigid body
- **Rotation along a fixed axis**
  - Rotational kinematics
  - Rotational kinetic energy
  - **Moment of inertia**
  - Parallel-axis theorem
- Newton's second law for rotation
  - **Torque (about an axis)**
  - Newton's second law for rotation

Outline (continued)

- **Plane-parallel motion**
  - **Dynamics for plane-parallel motion**
  - **Pure rolling**
  - Rotation with slipping
- **Angular Momentum**
  - Torque about a point & angular momentum
  - Theorem of angular momentum
  - Theorem of angular momentum for a system
  - **Conservation of angular momentum**

Outline (continued)

- **Plane-parallel motion**
  - **Dynamics for plane-parallel motion**
  - **Pure rolling**
  - Rotation with slipping
- **Angular Momentum**
  - Torque about a point & angular momentum
  - Theorem of angular momentum
  - Theorem of angular momentum for a system
  - **Conservation of angular momentum**
- Equilibrium
  - Conditions for equilibrium
  - Static equilibrium in an accelerated frame
  - Stability of rotational equilibrium

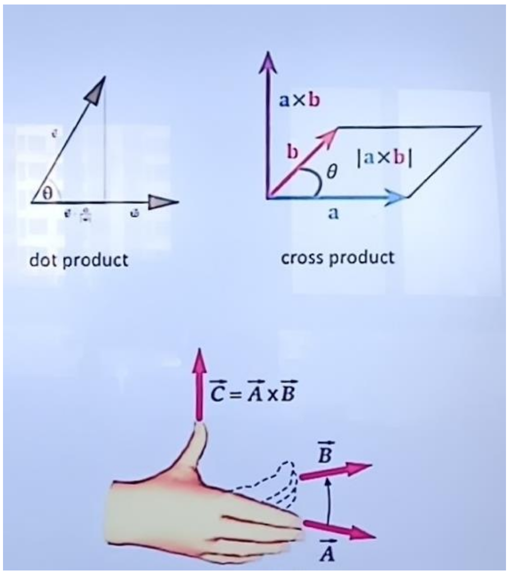
Vector Product 矢量叉乘

Vector Product (Cross Product) 叉乘:

Magnitude:

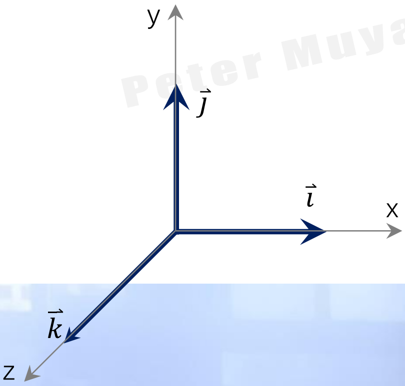
$$|\vec{a} \times \vec{b}| = ab\,sin\,\theta$$

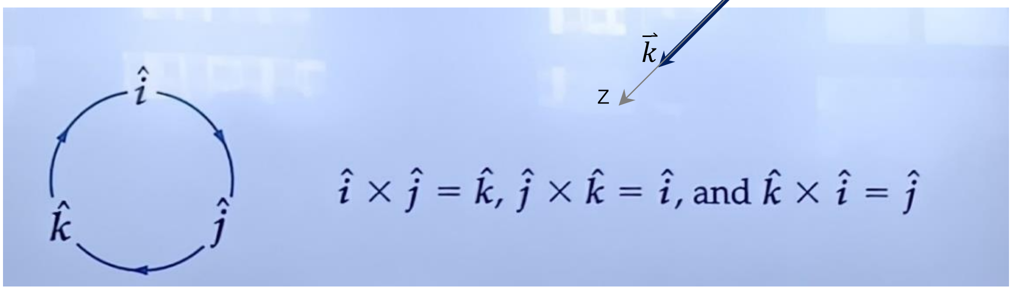
Direction(Right hand rule):
rotate from the first vector to the second vector.



Right-handed coordinate system:

$$\vec{\imath} \times \vec{\jmath} = \vec{k}$$





Vector Product calculation:

$$\vec{a} \times \vec{b} = (a_x\vec{\imath} + a_y\vec{\jmath} + a_z\vec{k}) \times (b_x\vec{\imath} + b_y\vec{\jmath} + b_z\vec{k})$$

$$= (a_yb_z - a_zb_x)\vec{\imath} + (a_zb_x - a_xb_z)\vec{\jmath} + (a_xb_y - a_yb_x)\vec{k}$$

Simplified notation:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

Properties:

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$
$$\vec{a} \times \vec{a} = 0$$
$$\frac{d}{dt}(\vec{a} \times \vec{b}) = \frac{d\vec{a}}{dt} \times \vec{b} + \vec{a} \times \frac{d\vec{b}}{dt}$$
$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \times \vec{b} - (\vec{a} \cdot \vec{b}) \times \vec{c}$$

Rigid Body 刚体

Rigid body:

In order to investigate the problem. We need to make proper simplification.

Recap: mass point

A point with mass but no spatial extent.

Condition of an object can be regard as a mass point:

The object's size is not significant to the problem.

Can we simplify the objects as mass point in the rotational case?

Ideal model: Rigid body.

Rigid body: is a solid body in which deformation is zero or so small it can be neglected. The distance between any two given points on a rigid body remains constant in time regardless of external forces or torques exerted on it.

Examples of rigid body:

Condition of an object can be regard as rigid body: The deformation of the object can be neglected.

Property of rigid body:

- The total work done by the internal force in a rigid body is always zero.

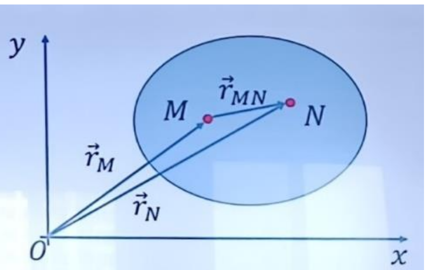
We don't need to consider the work done by the internal force when we apply work-energy principle to the rigid body

Rigid body (Optional)

For the point M and N in a rigid body. The displacement of M and N is $d\vec{r}_M$ and $d\vec{r}_N$, internal force of M act on N is $\vec{F}_{MN}$

Then $\vec{F}_{MN} = -\vec{F}_{NM}$

Total work done by internal force



$$dW = \vec{F}_{MN} d\vec{r}_N + \vec{F}_{NM} d\vec{r}_M = \vec{F}_{MN}(d\vec{r}_N - d\vec{r}_M) = \vec{F}_{MN} d(\vec{r}_N - \vec{r}_M)$$

Property of rigid body:

$$|\vec{r}_{MN}| = |\vec{r}_N - \vec{r}_M| = \text{const} \Longrightarrow d(\vec{r}_{MN}^{\,2}) = 2(\vec{r}_N - \vec{r}_M)d(\vec{r}_N - \vec{r}_M) = 0$$

By newton's third law:

$$\vec{F}_{MN} \parallel (\vec{r}_N - \vec{r}_M) \Longrightarrow \vec{F}_{MN} d(\vec{r}_N - \vec{r}_M) = 0 \Longrightarrow dW = 0$$

Degrees of freedom of Rigid body:

Degree of freedom: an independent physical parameter in the formal description of the state of a physical system.

Degrees of freedom in mechanical: the number of **independent** parameters needed to identify the location of the system.

$i=1$ $\qquad$ $i=2$ $\qquad$ $i=3$

Degrees of freedom of Rigid body:

A system contains N particles should have a degrees of freedom of $3N$.

But there are some conditions (constraints) that will reduce the number of independent parameters.

One constraint

$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2=r^2$$

Degrees of freedom of Rigid body:

A system contains N particles should have a degrees of freedom of $3N$.

But there are some conditions (constraints) that will reduce the number of independent parameters.

One constraint

$$(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2=r^2$$

$$i=6-1=5$$

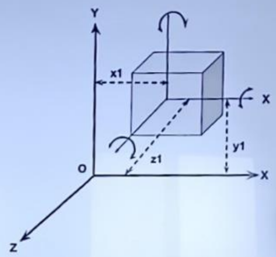
Two more constraints

$$i=9-1-2=6$$

Degrees of freedom of Rigid body:

For a rigid body without any constraint, the degrees of freedom is **6**.

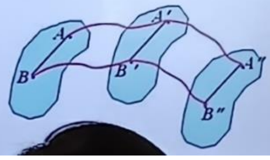
- Three translational degrees of freedom.
- Three rotational degrees of freedom.



Some typical motion of rigid body.

- Translational motion ($i = 3$):
  Any segment connecting two points in the rigid body remains parallel during the motion.

  It can be regards as a mass point.



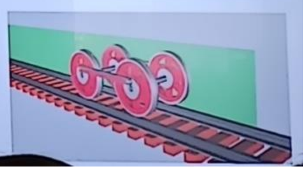
Degrees of freedom of Rigid body:

- Rotation along a fixed axis ($i = 1$):
  Two points in the rigid body are fixed.

  One rotational degree of freedom

- Plane-parallel motion ($i = 3$):
  Every point in the rigid body moves parallelly to a certain plane.

  One rotational degree of freedom
  Two translational degrees of freedom



Degrees of freedom of Rigid body:

- Fixed-point motion ($i = 3$)

  Three rotational degrees of freedom

- General motion ($i = 6$):
  Can be decomposed into translational motion of a point and rotation around the point.

  We usually choose the center of mass as the reference point

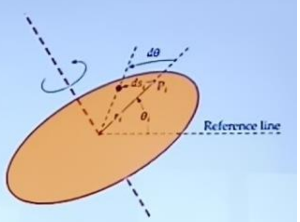
Rotation along a fixed axis

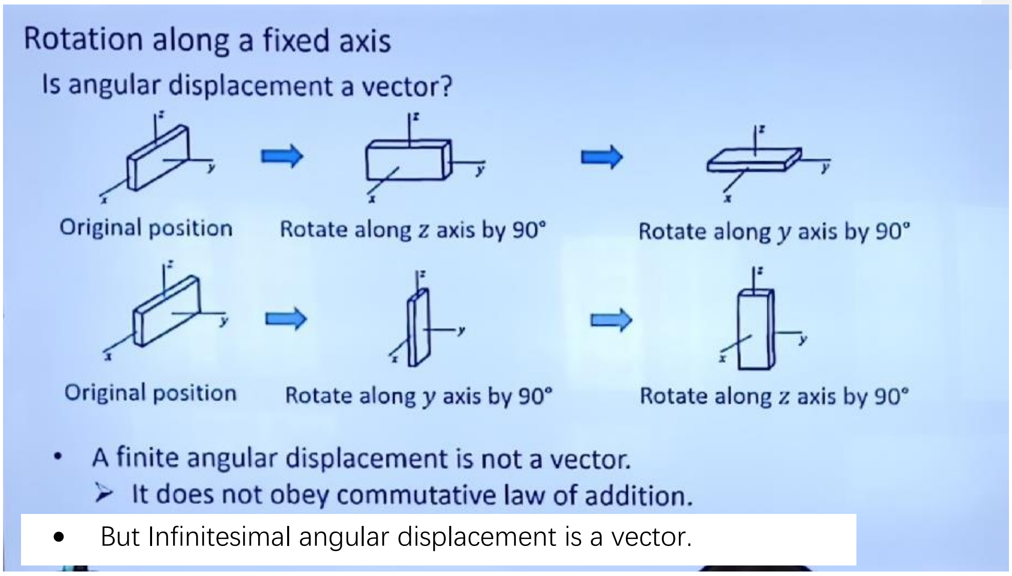
Only one rotational degree of freedom

Angular displacement: $d\theta = \frac{ds_i}{r_i}$

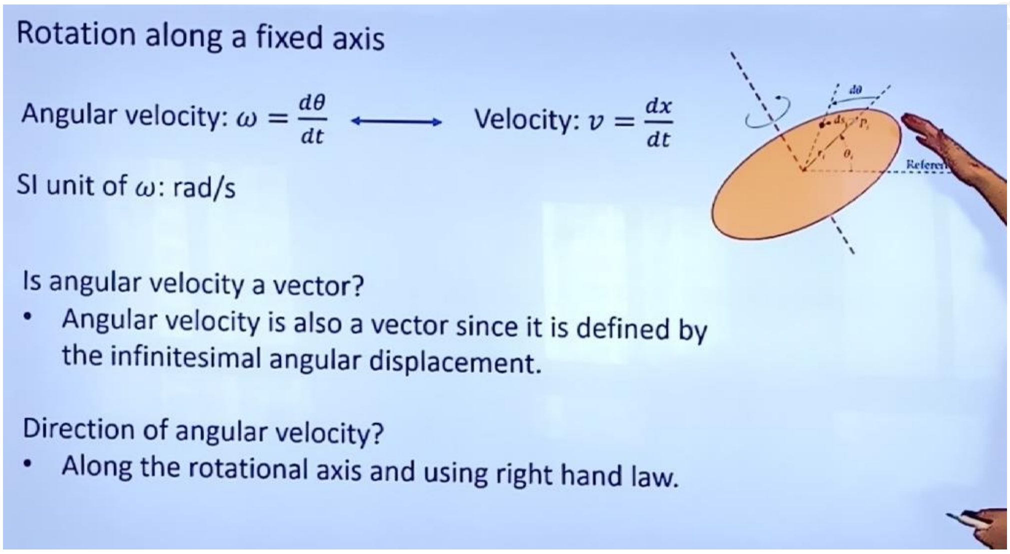
Conversion relationship: 1rev = 2π rad = 360°

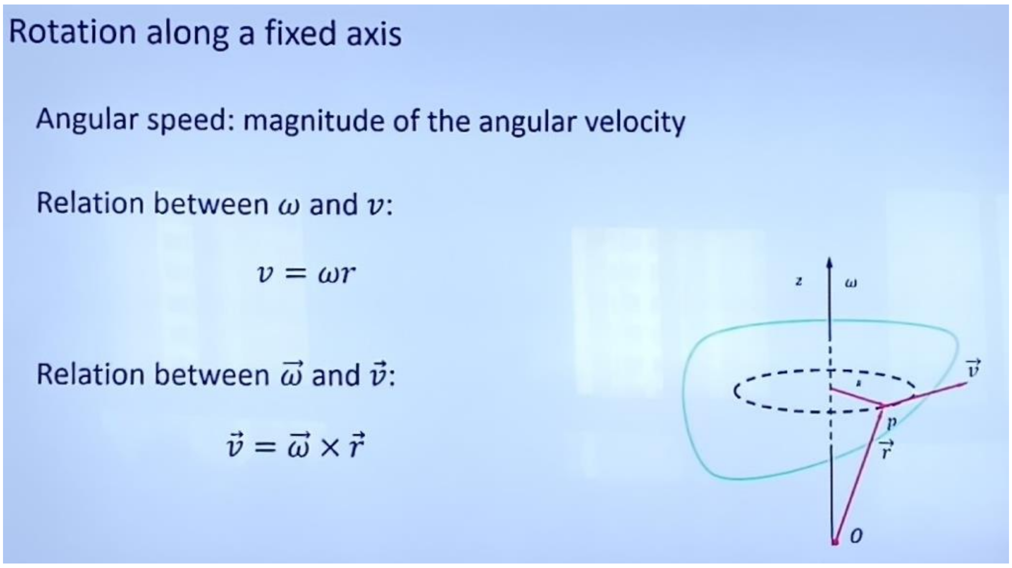
$d\theta$ keeps the same for all point on a rotational object.

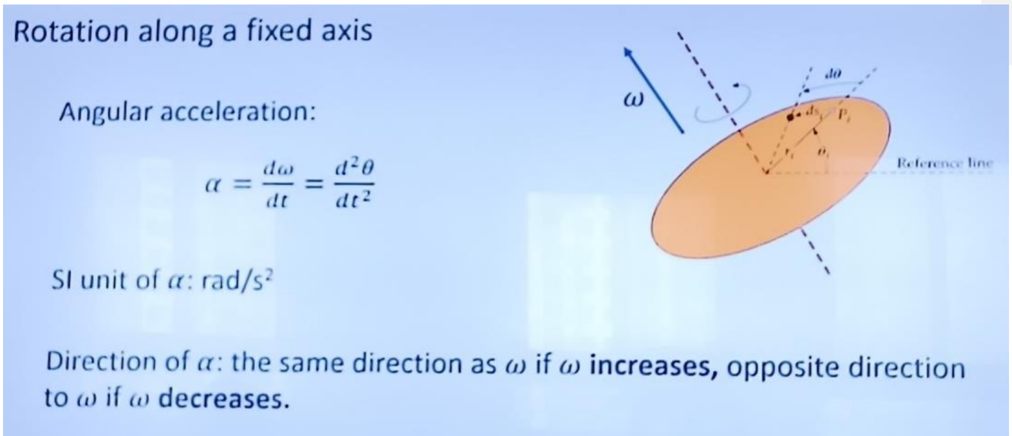




Rotation along a fixed axis:







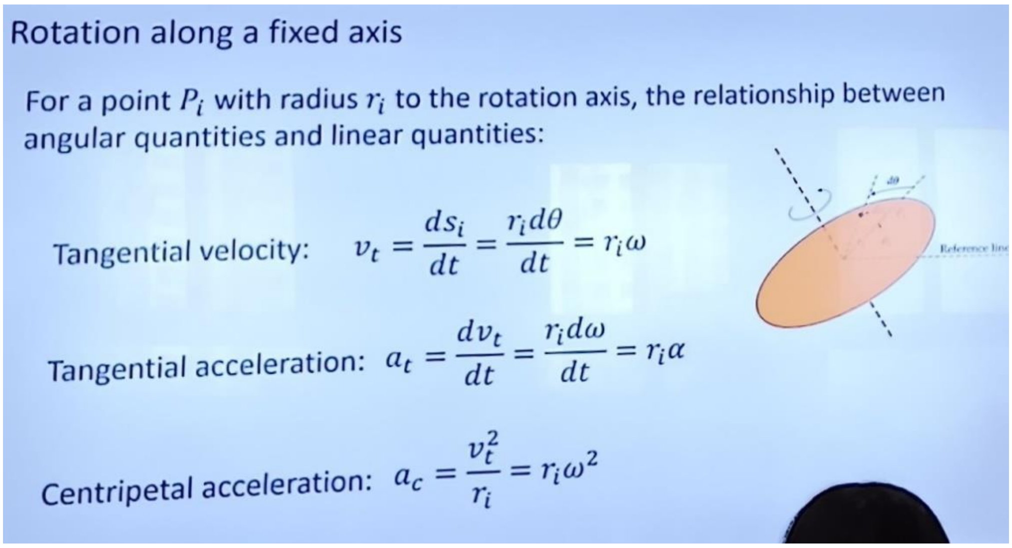
$$\alpha = \frac{d\omega}{dt} \Rightarrow d\omega = \alpha dt \ \Rightarrow \ \omega - \omega_0 = \int \alpha dt$$

If α is constant, then:

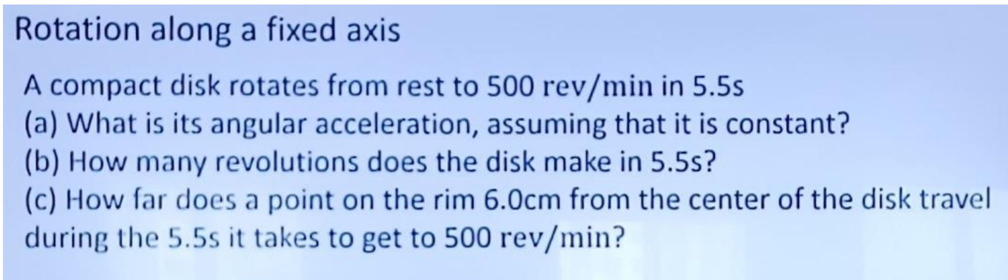
$$\omega = \omega_0 + \alpha t \qquad v = v_0 + at$$

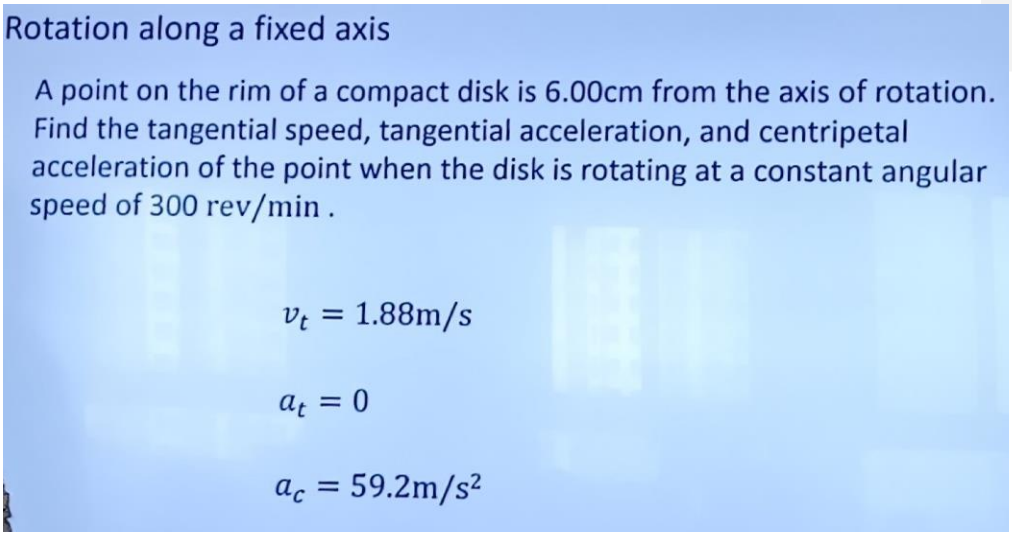
$$\theta = \theta_0 + \omega t + \frac{1}{2}\alpha t^2 \qquad x = x_0 + vt + \frac{1}{2}at^2$$

$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0) \qquad v^2 = v_0^2 + 2\alpha(x - x_0)$$



Example

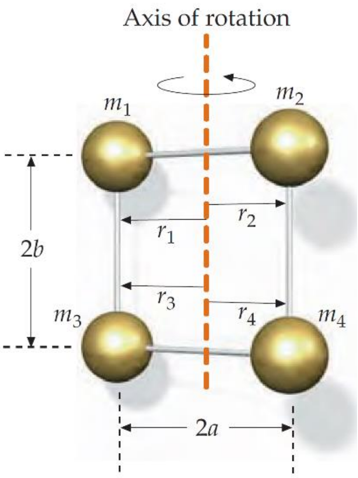




Rotational kinetic energy

Particle system	Rotation system
Mass m	Moment of inertia I Discrete particles: $I = \sum m_i r_i^2$ Continuous object: $I = \int r^2 dm$
Kinetic energy $E_k = \frac{1}{2}mv^2$	Kinetic energy $E_k = \frac{1}{2}I\omega^2$

Example: An object consists of four point particles, each of mass m, connected by rigid massless rods to form a rectangle of edge lengths 2a and 2b, as shown in the right figure. The system rotates with angular speed ω about an axis in the plane of the figure through the center, as shown.



(a) Find the kinetic energy of this object using $E_k = \frac{1}{2}I\omega^2$.

$$I = \sum m_i r_i^2 = mr_1^2 + mr_2^2 + mr_3^2 + mr_4^2$$

$$= ma^2 + ma^2 + ma^2 + ma^2 = 4ma^2$$

$$E_k = \frac{1}{2}I\omega^2 = \frac{1}{2}(4ma^2)\omega^2 = 2ma^2\omega^2$$

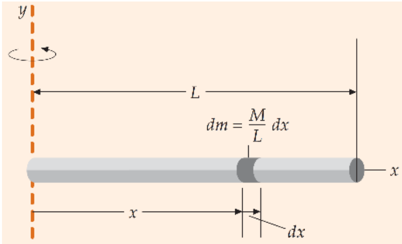
(b) Check your result by individually calculating the kinetic energy of each particle and then taking their sum.

Speed of each particle: $v_i = r_i\omega = a\omega$

Kinetic energy for each one: $E_{ki} = \frac{1}{2}mv_i^2 = \frac{1}{2}ma^2\omega^2$

Total kinetic energy: $E_k = 4(\frac{1}{2}ma^2\omega^2) = 2ma^2\omega^2$

Example: Find the moment of inertia of a thin uniform rod of length L and mass M about an axis perpendicular to the rod and through one end.



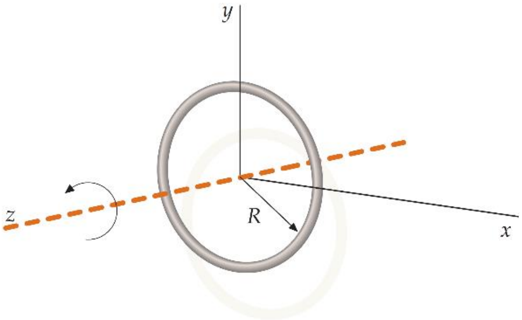
1. The linear density of the rod is: λ=M/L
2. A mass element dm at a distance x from the axis.

$$dm = \lambda dx = \frac{M}{L}dx$$

3. The moment of inertia about the y axis is:

$$I = \int x^2 dm = \int_0^L x^2 \frac{M}{L}dx = \frac{M}{L}\int_0^L x^2 dx = \frac{M}{L}\left(\frac{x^3}{3}\right)_0^L = \frac{1}{3}ML^2$$

Example: Find the moment of inertia of a hoop of mass M and radius R. The axis of rotation is the symmetry axis of the hoop, which is perpendicular to the plane of the hoop and through its center.

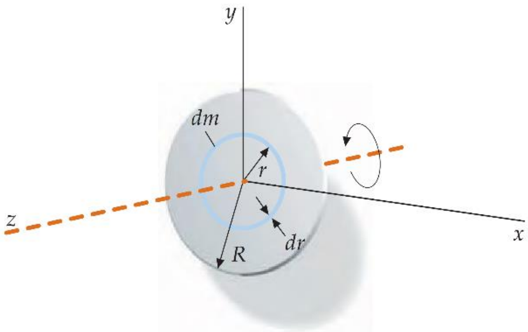


$$I = \int r^2 dm$$

Please note that the radius r is independent of dm, so the equation above can be:

$$I = r^2 \int dm = R^2 \int dm = R^2 M$$

Example: Find the moment of inertia of a uniform disk about a perpendicular axis through its center. The disk has mass M and radius R.



1. The mass per unit area σ is:

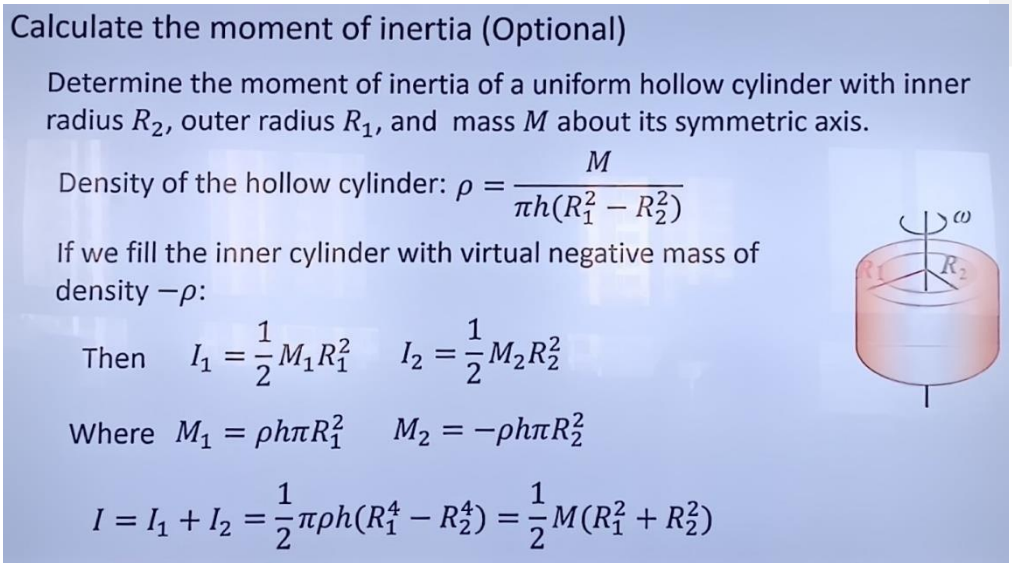
$$\sigma = \frac{M}{A} = \frac{M}{\pi R^2}$$

2. For the mass element of a hoop with radius r, thickness dr and mass dm:

$$dm = \sigma dA = \frac{M}{\pi R^2}(2\pi r dr)$$

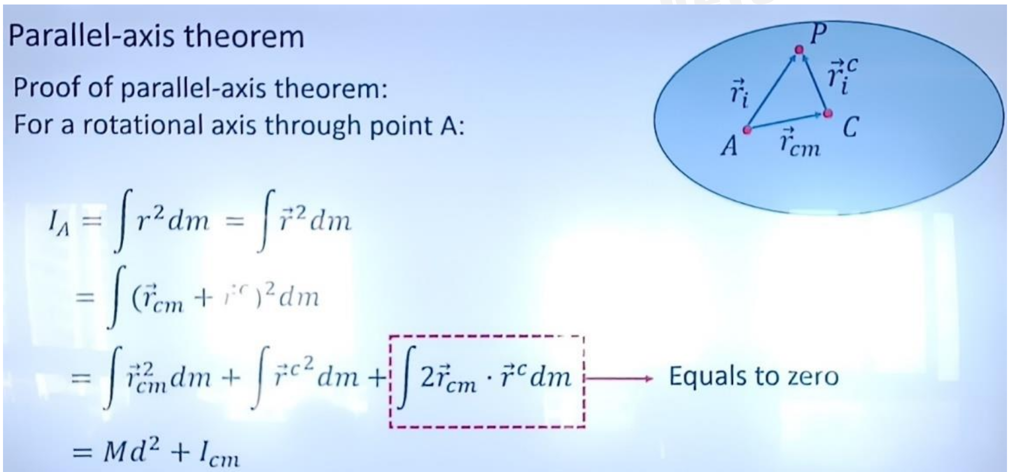
3. The moment of inertia:

$$I = \int r^2 dm = \int_0^R r^2 \frac{M}{\pi R^2} 2\pi r dr = \frac{2M}{R^2}\int_0^R r^3 dr = \frac{2M}{R^2}\left(\frac{r^4}{4}\right)_0^R = \frac{1}{2}MR^2$$

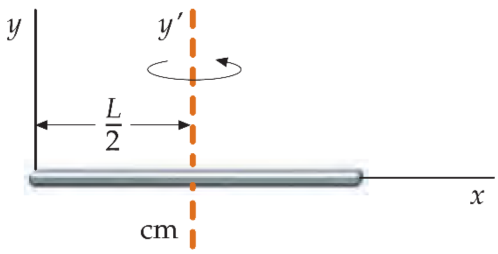


Parallel-axis theorem

$$I_A = Md^2 + I_{cm}$$
$$d = r_{cm}$$



Example: Applying the Parallel-Axis Theorem.
A thin uniform rod of mass M and length L on the x axis has one end at the origin. Using the parallel-axis theorem, find the moment of inertia about the y' axis, which is parallel to the y axis, and through the center of the rod.



1. We have got that $I_y = \frac{1}{3}ML^2$ about one end and want to find I_{cm}

2. Use the parallel-axis theorem with $d = \frac{1}{2}L$

$$I_y = Md^2 + I_{cm} \Rightarrow I_y = M(\frac{1}{2}L)^2 + I_{cm} \Rightarrow I_{cm} = I_y - \frac{1}{4}ML^2 = \frac{1}{12}ML^2$$

Newton's 2nd Law for Rotation 旋转牛二

A single force:

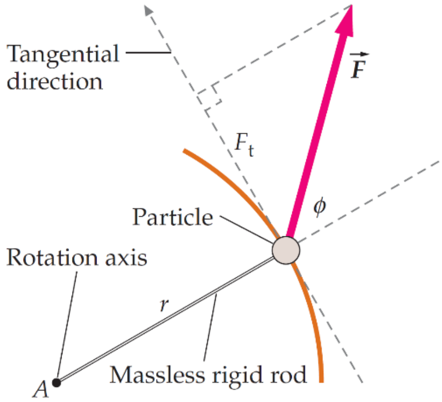
$\vec{F}$

The tangential direction component if it is:

$F_t = F\sin\emptyset$

$F_t = ma_t$

$rF_t = rma_t = rm(r\alpha) = mr^2\alpha$



Definition: **Torsional torque or torque 扭矩 τ**

$\tau = F_t r = mr^2\alpha$

SI unit: N·m

Direction: along the rotational axis and using right hand law.

For rigid object that rotate about a fixed axis

$\tau_{i\,net} = m_i r_i^2\alpha$

$$\sum \tau_{i\,net} = \sum m_i r_i^2\alpha = \left(\sum m_i r_i^2\right)\alpha = I\,\alpha$$

Newton's second law for Rotation

$$\sum \tau_{i\,net} = \sum \tau_{ext} = I\,\alpha$$

Can be rewrite as

$$\tau = I\,\alpha = Ira$$

Notice

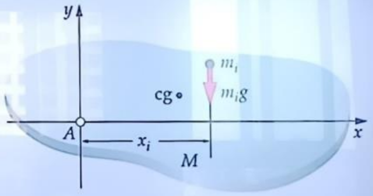
1. Torque here means the torque about the rotation axis.
2. The new torque caused by internal force always equals to zero.

Torque due to gravity

The net gravitational torque can be calculated by considering the total gravitational force (the sum of the microscopic gravitational forces) to act at a single point $i.e.$ **center of gravity.**

If the gravitational field $\vec{g}$ is parallel.

$$\vec{r}_{cg} = \frac{\sum_{i=1}^{n} w_i \vec{r}_i}{\sum_{i=1}^{n} w_i} = \frac{\sum_{i=1}^{n} m_i g_i \vec{r}_i}{\sum_{i=1}^{n} m_i g_i}$$



If the gravitational field $\vec{g}$ is uniform.

$$\vec{r}_{cg} = \frac{\sum_{i=1}^{n} w_i \vec{r}_i}{\sum_{i=1}^{n} w_i} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i} = \vec{r}_{cm}$$

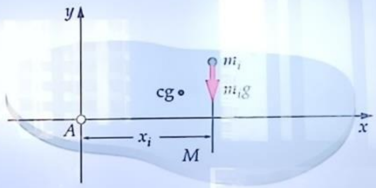
The difference between cg and cm?

Torque due to gravity

The net gravitational torque can be calculated by considering the total gravitational force (the sum of the microscopic gravitational forces) to act at a single point $i.e.$ **center of gravity.**

$$\tau_{grav} = F_g r_{cg} \sin\theta$$

If the gravitational field $\vec{g}$ is uniform and let x-axis perpendicular to $\vec{g}$



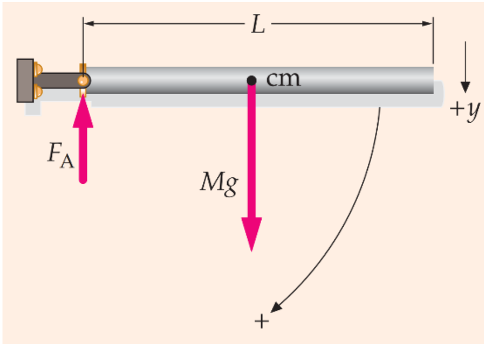
$$\tau_{grav} = Mgx_{cg}$$
$$= \sum_i x_i m_i g$$

Applying Newton's second law for Rotation along a fixed axis:

Procedure:

1. Draw a free-body diagram with the object shown as a likeness of object (not just as a dot).
2. Draw each force vector along the line of action of that force.
3. On the diagram indicate the positive direction (clockwise or counterclockwise) for rotation.

Example: A uniform thin rod of length L and mass M is pivoted at one end. It is held horizontal and released. Effects due to friction and air resistance are negligible. Find



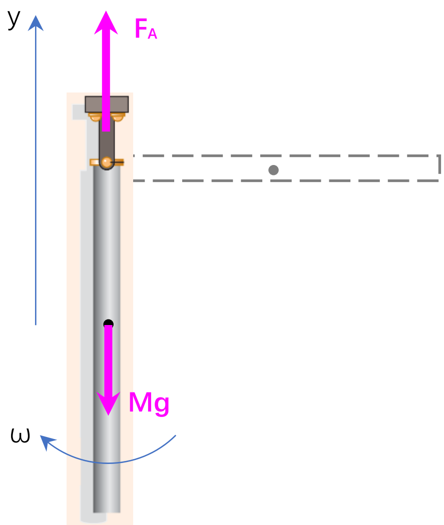
(a) the angular acceleration of the rod immediately following its release, and

1. Sketch a free-body diagram of the rod shown above:
2. Write Newton's second law for rotation: $\sum \tau_{ext} = I\,\alpha$
3. Compute the torque due to gravity about the given axis. The rod is uniform, so its center of gravity is at its center, a distance L/2 from the axis: $\tau_{grav} = Mg\frac{L}{2}$
4. Find the moment of inertia about the end of the rod: $I = \frac{1}{3}ML^2$
5. Substitute these values into the step-2 equation to compute α:

$$\alpha = \frac{\tau_{grav}}{I} = \frac{Mg\frac{L}{2}}{\frac{1}{3}ML^2} = \frac{3g}{2L}$$

注意I是关于端点的，不是关于CM的

(b) the magnitude of the force exerted on the rod by the pivot at this instant.



1. Sketch a free-body diagram of the rod:
2. Write Newton's second law for a system for the rod: $F_A - Mg = Ma_{cm\,y}$
3. Conservation of Mechanical energy:

$$Mg\left(\frac{L}{2}\right) = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{3}ML^2\right)\omega^2$$

$$\omega^2 = \frac{3g}{L}$$

4. Use the relation $a_n = r\omega^2$ to find a_n,:

$$a_{cm\,n} = r_{cm}\omega^2 = \frac{L}{2}\omega^2 = \frac{L}{2}\frac{3g}{L} = \frac{3}{2}g$$

5. $F_A = Mg + Ma_{cm\,n} = Mg + M\left(\frac{3}{2}g\right) \Rightarrow F_A = \frac{5}{2}g$

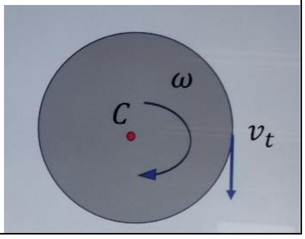
(c) What initial angular speed would be needed for the rod to just reach vertical position at the top of its swing?

$$Mg\left(\frac{L}{2}\right) = \frac{1}{2}I\omega^2 = \frac{1}{2}\left(\frac{1}{3}ML^2\right)\omega^2 \Rightarrow \omega = \sqrt{\frac{3g}{L}}$$

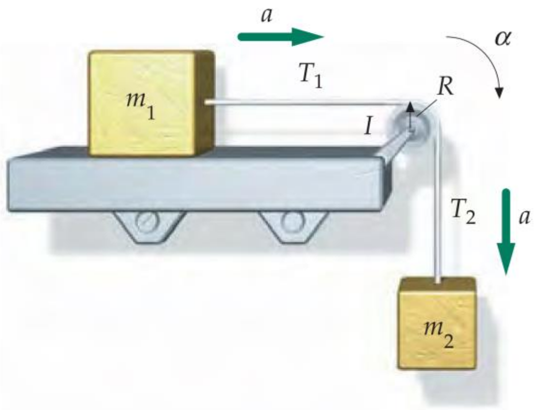
Nonslip condition of string on a pulley wheel:

$$v_t = \omega R$$

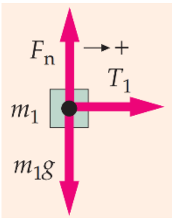
$$a_t = \alpha R$$

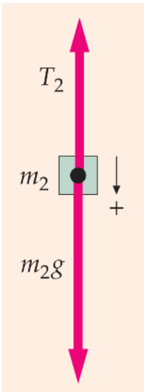


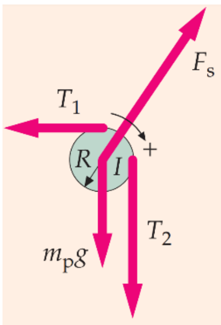
Example: Two blocks are connected by a string that passes over a pulley of radius R and moment of inertia I. The block of mass m_1 slides on a frictionless, horizontal surface; the block of mass m_2 is suspended from the string. Find the acceleration α of the blocks and the tensions T_1 and T_2. The string does not slip on the pulley.



1. Free-body diagram of block m_1
   $T_1 = m_1 a$
   Free-body diagram of block m_2
   $m_2 g - T_2 = m_2 a$
   Free-body diagram of the pulley
   $T_2 R - T_1 R = I\alpha$







2. For non-slip condition: $a = a_t = \alpha R$

3. $a = \frac{m_2}{m_1+m_2+\left(\frac{I}{R}\right)^2} g$

$$T_1 = m_1 a = \frac{m_1 m_2}{m_1 + m_2 + \left(\frac{I}{R}\right)^2} g$$

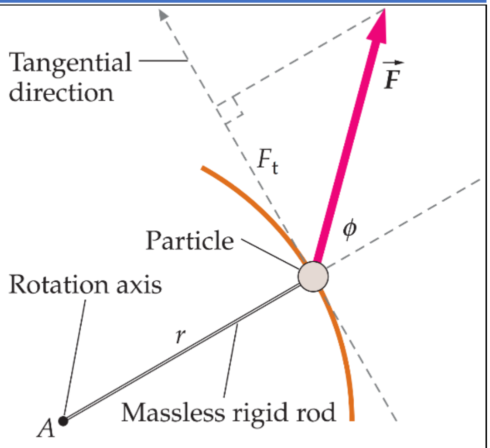
$$T_2 = m_2(g-a) = \frac{m_1 + \left(\frac{I}{R}\right)^2}{m_1 + m_2 + \left(\frac{I}{R}\right)^2} m_2 g$$

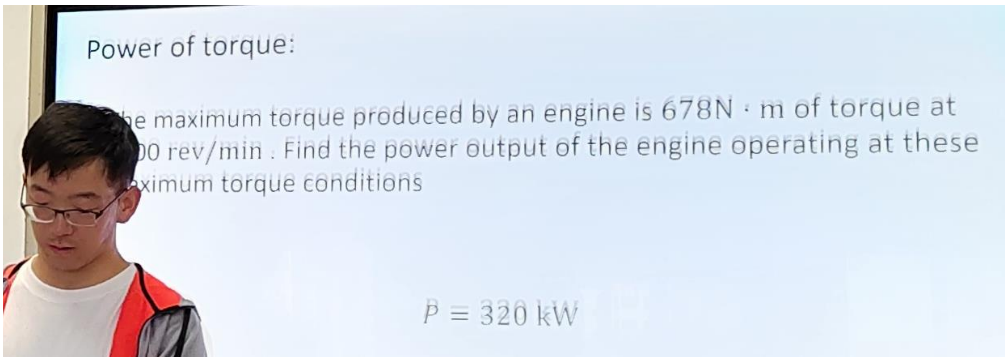
work of torque:

$$dW = \vec{F} d\vec{l}$$
$$= F_t ds$$
$$= F_t r d\theta$$
$$= \tau d\theta$$

Power of torque:

$$P = \frac{dW}{dt}$$
$$= \tau \frac{d\theta}{dt}$$
$$= \tau\omega$$





Plane-parallel motion 平面平行运动

Every point in the rigid body moves parallelly to a certain plane.

Dynamic equations of plane-parallel motion:

Translational motion of center of mass:

$$\sum \vec{F}_{net} = M\vec{a}_{cm}$$

Rotation along the center of mass:

$$\tau_{ext-cm} = I_{cm}\alpha$$

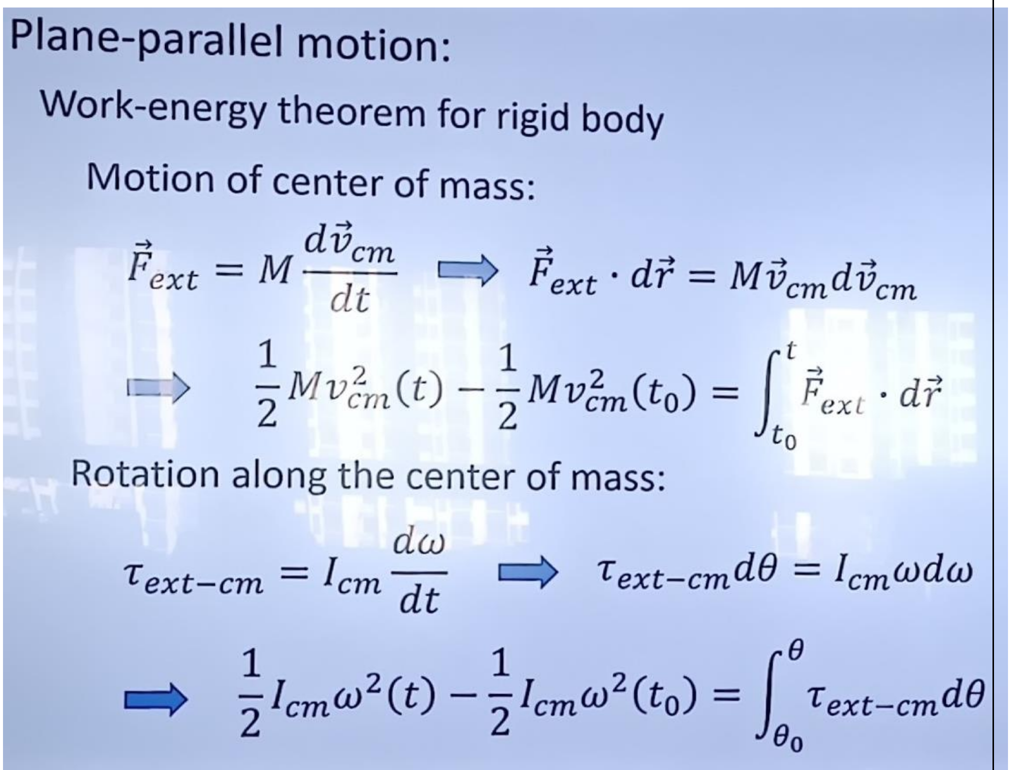
- The net torque caused by inertial force in CM frame to the entire system is always zero.
- **Even if $\vec{a}_{cm} \neq 0$,** the equation above is correct.

Plane-parallel motion can be determined by:

$$\begin{cases} F_{ext-x} = Ma_{cm-x} \\ F_{ext-y} = Ma_{cm-y} \\ \tau_{ext-cm} = I_{cm}\alpha \end{cases}$$

Kinetic Energy:

$$E_k = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$$ Only in CM reference fame

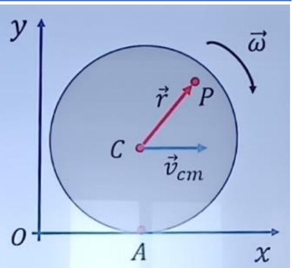


Work-energy theorem for rigid body

$$\Delta E_k = E_k(t) - E_k(t_0) = \int_{t_0}^{t} \vec{F}_{ext} \cdot d\vec{r} + \int_{\theta_0}^{\theta} \tau_{ext-cm} \cdot d$$

The change in kinetic energy of a rigid body is equal to the work done by external forces on its **center of mass** plus the work done by external torques about the **center of mass**.

Rotation without slipping (pure rolling):



If we take CM as the reference point, and the velocity of CM is $\vec{v}_{cm}$, angular velocity of the rigid body is $\vec{\omega}$, then for each point P with position $\vec{r}$ on the rigid body

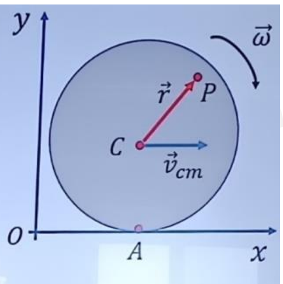
$$\vec{v}_P = \vec{v}_{cm} + \vec{\omega} \times \vec{r}$$

If the velocity of the contact point A is zero, it rotates without slipping.

$$\vec{v}_A = \vec{v}_{cm} + \vec{\omega} \times \vec{r}_A = (v_{cm} - \omega R)\hat{\imath} = 0$$

Nonslip condition: $v_{cm} = \omega R$ ⇨ Relation between acceleration: $a_{cm} = \alpha R$

Rotation without slipping (pure rolling):



Since the velocity of point A is zero. The rigid body rotates along the axis through point A at this moment.

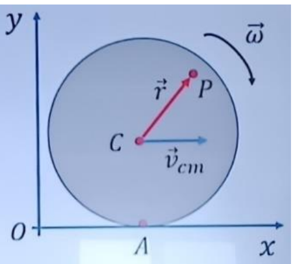
The point A is called the **instantaneous center (of rotation) or instantaneous center of velocity.**

- Is the acceleration of point A equals to zero?

$$\vec{a}_A = \vec{a}_{cm} + \vec{\alpha} \times \vec{r}_A - \omega^2 \vec{r}_A = -\omega^2 \vec{r}_A \Longrightarrow \vec{a}_A \neq 0$$

The point P with $\vec{a}_P = 0$ is called the instantaneous center of acceleration

Rotation without slipping (pure rolling) (Optional)



Can we take point A as reference point? *Yes*

Benefit of we take point A as reference point:

The motion of the rigid body is fixed axis rotation.

Problems need to be care about:

- $\vec{a}_A \neq 0$ means the reference frame is non-inertial, we need to consider the inertial torque generated by inertial force.
  - ➢ If the mass distribution is uniform, then $\tau_{iner-A} = 0$
  - ➢ Then $\tau_{ext-A} = I_A \alpha$
- Does the angular velocity the same about center of mass and point A, i.e. $\vec{\omega}_A = \vec{\omega}_{CM}$?

Angular velocity of the rigid body about an arbitrary point (Optional)

Take O_2 as reference point:

$$\vec{v} = \frac{d\vec{R}}{dt} = \frac{d(\vec{R}_2 + \vec{r}_2)}{dt} = \vec{v}_2 + \frac{d\vec{r}_2}{dt} = \vec{v}_2 + \vec{\omega}_2 \times \vec{r}_2$$

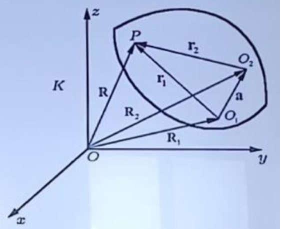
$\Rightarrow$ $\vec{\omega}_1 \times \vec{r}_1 = \vec{\omega}_1 \times \vec{a} + \vec{\omega}_2 \times \vec{r}_2$

Geometric relationship: $\vec{r}_1 = \vec{a} + \vec{r}_2$

$\Rightarrow$ $\vec{\omega}_1 \times \vec{r}_2 = \vec{\omega}_2 \times \vec{r}_2$

Since O_1, O_2, P are three arbitrary points on the rigid body.

$\Rightarrow$ $\vec{\omega}_1 = \vec{\omega}_2$



Angular velocity of the rigid body about an arbitrary point (Optional)

Take O_2 as reference point:

$$\vec{v} = \frac{d\vec{R}}{dt} = \frac{d(\vec{R}_2 + \vec{r}_2)}{dt} = \vec{v}_2 + \frac{d\vec{r}_2}{dt} = \vec{v}_2 + \vec{\omega}_2 \times \vec{r}_2$$

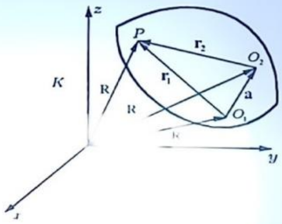
$\Rightarrow$ [illegible] $\times \vec{r}_1 =$ [illegible] $\times \vec{r}_2$

[illegible]etric relati[illegible] $= \vec{a} + \vec{r}_2$

$\Rightarrow$ $\vec{\omega}_1 \times \vec{r}_2 = \omega_2 \times r_2$

Since O_1, O_2, P are three arbitrary points on the rigid body.

$\Rightarrow$ $\vec{\omega}_1 = \vec{\omega}_2$

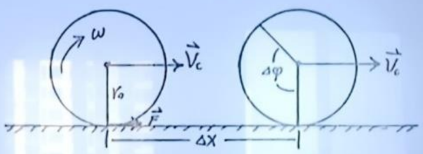


Angular velocity of a rigid body is irrelevant to the reference point.

Rotation without slipping (pure rolling):

Work of friction in the pure rolling:

$$
\begin{aligned}
W &= \int \vec{F} d\vec{r} + \int \tau d\varphi \\
&= \int F dx + \int -F r_0 d\varphi \\
&= F\left(\int dx - \int r_0 d\varphi\right) \\
&= F(\Delta x - r_0 \Delta\varphi) \\
&= 0
\end{aligned}
$$



Rotation without slipping (pure rolling):

A sphere of mass M, radius r, and rotational inertia I is released from rest at the top of an inclined plane of height h as shown, If the plane has friction so that the sphere rolls without slipping, what is the speed v_{cm} of the center of mass at the bottom of the incline?

$$v_{cm} = \sqrt{\frac{2Mghr^2}{I + Mr^2}}$$

Plane-parallel motion:

Rotation along the center of mass:

$$\tau_{ext-cm} = I_{cm}\alpha$$

Do we need to modify the above equation if $\vec{a}_{cm} \neq 0$?

- The net torque caused by inertial force in CM frame to the entire system is always zero.
- **Even if $\vec{a}_{cm} \neq 0$, the above equation is correct.**

$$\boxed{\begin{aligned} F_{ext-x} &= Ma_{cm-x} \\ F_{ext-y} &= Ma_{cm-y} \\ \tau_{ext-cm} &= I_{cm}\alpha \end{aligned}}$$

Plane-parallel motion can be determined by these three equations

Plane-parallel motion:

Work-energy theorem for rigid body

- Kinetic Energy for rigid body:

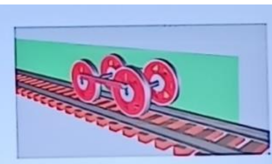
$$E_k = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$$

- Work-energy theorem for rigid body:

$$\Delta E_k = E_k(t) - E_k(t_0) = \int_{t_0}^{t} \vec{F}_{ext} \cdot d\vec{r} + \int_{\theta_0}^{\theta} \tau_{ext-cm} d\theta$$

The change in kinetic energy of a rigid body is equal to the work done by external forces on its **center of mass** plus the work done by external torques about the **center of mass**.

Plane-parallel motion:



- Kinetic Energy:

$$E_k = \sum_i \frac{1}{2}m_i v_i^2 \Rightarrow \text{General case}$$

$$\boxed{\begin{aligned} E_k &= \frac{1}{2}Mv_{cm}^2 + \sum_i \frac{1}{2}m_i v_i^{c2} \\ E_k &= \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 \end{aligned}} \Rightarrow \text{Only in CM reference frame}$$

7. Non-slip Condition

When a string that is wrapped around a pulley or disk does not slip

$v_t = r\omega$

$a_t = r\alpha$

Rolling without slipping

$v_{cm} = r\omega$

$a_{cm} = r\alpha$

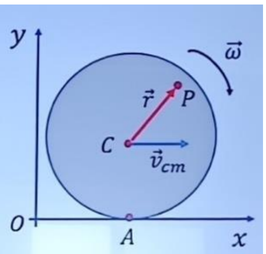
Point P with position $\vec{r}$ on the rigid body

$\vec{v}_p = \vec{v}_{cm} + \vec{\omega} \times \vec{r}$

If the velocity of the contact point A is zero

$\vec{v}_A = \vec{v}_{cm} + \vec{\omega} \times \vec{r}_A = (v_{cm} - \omega r)\vec{\imath} = 0$

Rotation without slipping (pure rolling) (Optional)



Can we take point A as reference point? *Yes*

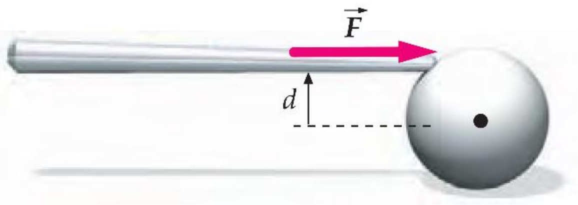
Benefit of we take point A as reference point:

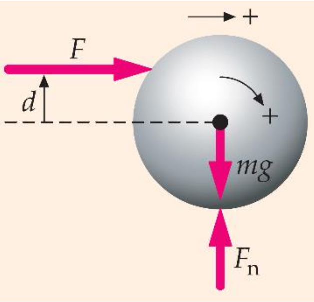
The motion of the rigid body is fixed axis rotation.

Problems need to be care about:

- $\vec{a}_A \neq 0$ means the reference frame is non-inertial, we need to consider the inertial torque generated by inertial force.
  - If the mass distribution is uniform, then $\tau_{iner-A} = 0$
  - Then $\tau_{ext-A} = I_A\alpha$
- Does the angular velocity the same about center of mass and point A, i.e. $\vec{\omega}_A = \vec{\omega}_{CM}$?

Example: A cue stick strikes a cue ball horizontally at a point a distance d above the center of the ball. Find the value of d for which the cue ball will roll without slipping from the beginning. Express your answer in terms of the radius R of the ball. ($I_{cm} = \frac{2}{5}MR^2$)





Newton's 2nd law:

$$\begin{cases} F = ma_{cm} & (1) \\ \tau = I_{cm}\alpha \\ \tau = Fd \end{cases}$$

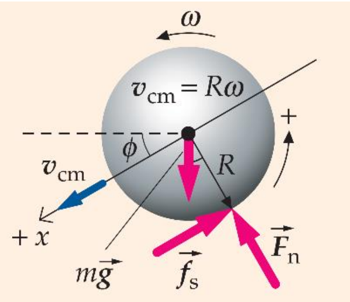
Nonslip condition:

$$a_{cm} = R\alpha$$

$$(1) \Rightarrow F = ma_{cm} = mR\alpha = mR\left(\frac{\tau}{I_{cm}}\right) = mR\left(\frac{Fd}{I_{cm}}\right)$$

$$\Rightarrow F = mR\left(\frac{Fd}{I_{cm}}\right) \Rightarrow 1 = mR\left(\frac{d}{I_{cm}}\right) \Rightarrow d = \frac{I_{cm}}{mR} = \frac{\frac{2}{5}MR^2}{mR} = \frac{2}{5R}$$

Example: A uniform solid ball of mass m and radius R rolls without slipping down a plane inclined at an angle Φ above the horizontal. Find the frictional force and the acceleration of the center of mass. ($I_{cm} = \frac{2}{5}MR^2$).



Newton's 2nd law:

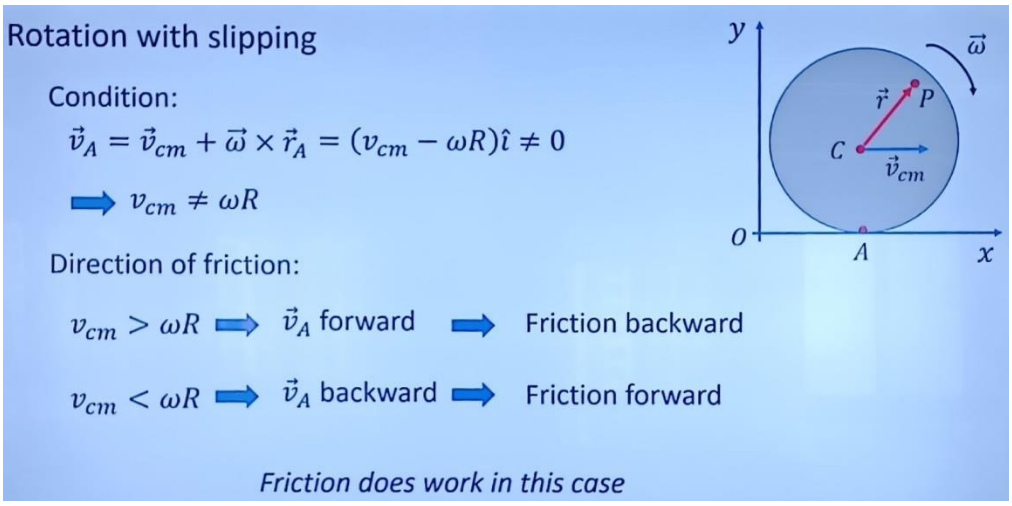
$$\begin{cases} F_x = mg\sin\Phi - f_s = ma_{cm-x} \\ \qquad \tau = f_s R = I_{cm}\alpha \end{cases}$$

Nonslip condition:

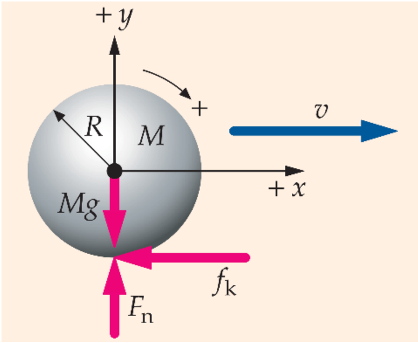
$$a_{cm} = R\alpha$$

$$\Rightarrow f_s = mg\sin\Phi - ma_{cm-x}$$
$$= mg\sin\Phi - mR\alpha$$
$$= mg\sin\Phi - mR(f_sR/I_{cm})$$
$$\Rightarrow f_s + mR(f_sR/I_{cm}) = mg\sin\Phi$$
$$\Rightarrow f_s[1 + mR^2/I_{cm}] = mg\sin\Phi$$
$$\Rightarrow f_s = mg\sin\Phi\,/(1 + mR^2/I_{cm}) = mg\sin\Phi\,/(1 + 5/2) = (2/7)\,mg\sin\Phi$$

Rotation with slipping (Optional)



Example: A bowling ball of mass M and radius R is released at floor level so that at release it is moving horizontally with speed $v_0 = 5.0\ m/s$ and is not rotating. The coefficient of kinetic friction between the ball and the floor is $\mu_k = 0.080$.
Find (a) the time the ball skids along the floor (after which it begins rolling without slipping), and (b) the distance the ball skids.



(a) 1. Sketch a free-body diagram of the ball (Figure 9-41).

2. The net force on the ball is the force of kinetic friction f_k, which acts in the negative x direction. Apply Newton's second law:

$$\Sigma F_x = Ma_{\text{cm}x}$$
$$-f_k = Ma_{\text{cm}x}$$

3. The acceleration is in the negative x direction and $a_{\text{cm}y} = 0$. Find f_k by first finding F_n:

$$\Sigma F_y = Ma_{\text{cm}y} = 0 \Rightarrow F_n = Mg$$
$$\text{so} \quad f_k = \mu_k F_n = \mu_k Mg$$

4. Find the acceleration using the step-2 and step-3 results:

$$-\mu_k Mg = Ma_{\text{cm}x} \Rightarrow a_{\text{cm}x} = -\mu_k g$$

5. Relate the linear velocity to the constant acceleration and the time using a kinematic equation:

$$v_{\text{cm}x} = v_0 + a_{\text{cm}x}t = v_0 - \mu_k g t$$

6. Find α by applying Newton's second law for rotational motion to the ball. Compute the torques about the axis through the center of mass. Note that the free-body diagram has clockwise as positive:

$$\Sigma\tau = I_{\text{cm}}\alpha$$
$$\mu_k MgR + 0 + 0 = \tfrac{2}{5}MR^2\alpha$$
$$\text{so} \quad \alpha = \frac{5}{2}\frac{\mu_k g}{R}$$

7. Relate the angular velocity to the constant angular acceleration and the time using a kinematic equation:

$$\omega = \omega_0 + \alpha t = 0 + \alpha t = \frac{5}{2}\frac{\mu_k g}{R}t$$

8. Solve for the time t at which $v_{\text{cm}} = R\omega$:

$$v_{\text{cm}} = R\omega$$
$$(v_0 - \mu_k g t) = R\left(\frac{5}{2}\frac{\mu_k g}{R}t\right)$$
$$\text{so} \quad t = \frac{2v_0}{7\mu_k g} = \frac{2(5.0\ \text{m/s})}{7(0.080)(9.81\ \text{m/s}^2)} = \boxed{1.8\ \text{s}}$$

(b) The distance traveled while skidding is

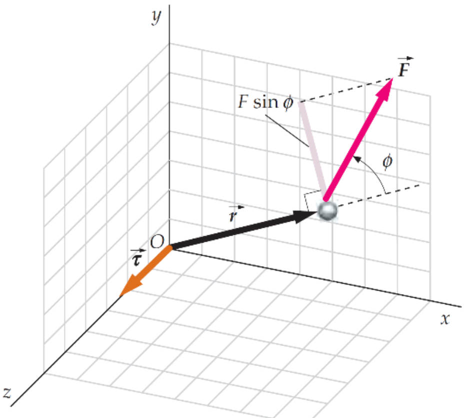
$$\Delta x = v_0 t + \tfrac{1}{2}a_{\text{cm}}t^2 = v_0\left(\frac{2v_0}{7\mu_k g}\right) + \frac{1}{2}(-\mu_k g)\left(\frac{2v_0}{7\mu_k g}\right)^2 = \frac{12}{49}\frac{v_0^2}{\mu_k g}$$
$$= \frac{12}{49}\frac{(5.0\ \text{m/s})^2}{(0.080)(9.81\ \text{m/s}^2)} = \boxed{7.8\ \text{m}}$$

Angular Momentum 角动量

Definition:

$\vec{L} = \vec{r} \times \vec{p}$

SI unit: $kg \cdot m^2/s$



Conversion:

$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$

$= \vec{r} \times m(\vec{\omega} \times \vec{r})$

$= mr^2\vec{\omega} - m(\vec{\omega} \cdot \vec{r})\vec{r}$

$\vec{L} = I\vec{\omega}$ only when the second term equals zero.

Angular momentum.

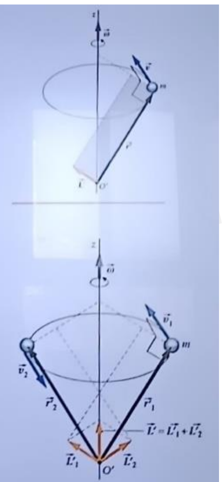
Usually, the angular momentum $\vec{L}$ and angular velocity $\vec{\omega}$ are not in the same direction

For a rigid body, if:

1. It is rotate along its' symmetry axis **or**
2. Its mass is distribute on a thin plane (thickness can be ignored), and it rotates along an axis that is perpendicular to the plane.

The angular momentum $\vec{L}$ and angular velocity $\vec{\omega}$ are in the same direction:

$\vec{L} = I\vec{\omega}$ where I is a scalar



Angular momentum.

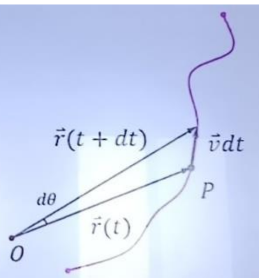
When an object moves in an inertial reference frame, it will rotate w.r.t. a certain point O.

During t to $t + dt$ the displacement of the object $d\vec{r} = \vec{v}dt$. The area of the segment OP swaps is

$$dA = \frac{1}{2}|\vec{r} \times \vec{v}dt|$$

Then the "area speed" is

$$\frac{dA}{dt} = \frac{1}{2}|\vec{r} \times \vec{v}| = \frac{|\vec{L}|}{2m}$$



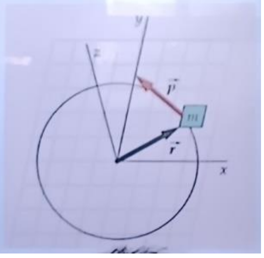
Angular momentum reflect the strength of the object rotate about a certain point.

Angular momentum.

Find the angular momentum about the origin for the following situations:

(a) A car of mass 1200kg moves in a circle of radius 20m with a speed of 15m/s The circle is in the xy plane, centered at the origin. When viewed from a point on the positive z axis, the car moves counterclockwise.

$$\vec{L} = \vec{r} \times \vec{p} = 3.6 \times 10^5 \text{kg} \cdot \frac{\text{m}^2}{\text{s}} \hat{k}$$

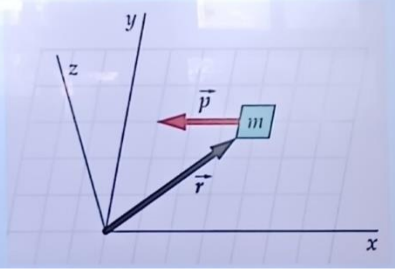


Angular momentum.

Find the angular momentum about the origin for the following situations:

(b) The same car (1200kg) moves in the xy plane with velocity $\vec{v} = -15\mathrm{m/s}\hat{\imath}$ along the line $y = y_0 = 20\mathrm{m}$ parallel to the x axis

$$\vec{L} = \vec{r} \times \vec{p} = 3.6 \times 10^5 \mathrm{kg} \cdot \frac{\mathrm{m}^2}{\mathrm{s}} \hat{k}$$



Theorem of angular momentum:

$$\vec{\tau}_{net} = \frac{d\vec{L}}{dt}$$

Net torque act on the object equals to the change rate of angular momentum of the object about the same point.

Integral representation:

$$\Delta\vec{L} = \int_0^t \vec{\tau}_{net} dt$$

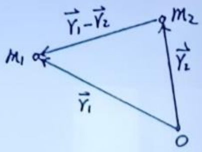
Angular impulse:

$$\vec{J} = \int_0^t \vec{\tau}_{net} dt \Rightarrow d\vec{L} = d\vec{J}$$

The change of angular momentum of the object equals to the angular impulse exert on the object about the same point.

Theorem of angular momentum for a system

Considering a isolated system consists of two particles:



Then $\dfrac{d\vec{L}_{sys}}{dt} = \dfrac{d(\vec{L}_1 + \vec{L}_2)}{dt} = \dfrac{d\vec{L}_1}{dt} + \dfrac{d\vec{L}_2}{dt}$

$$= \vec{r}_1 \times \vec{F}_{21} + \vec{r}_2 \times \vec{F}_{12}$$

$$= (\vec{r}_1 - \vec{r}_2) \times \vec{F}_{21}$$

$$= 0$$

$\Rightarrow$ $\vec{L}_{sys} = \vec{L}_1 + \vec{L}_2 = \text{constant}$

Total angular momentum of a two-particle isolated system about a certain point is conserved.

Theorem of angular momentum for a system

Notice:

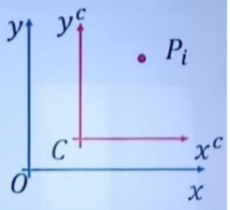
1. Theorem of angular momentum is only suitable for inertial reference frame
2. Angular momentum is determined by the state of the system and the **reference point**.
3. Angular impulse is related to the process and **the reference point**.
4. The reference point can be **any stationary point** in the inertial frame
5. The equation is a vector equation, the net external torque in any direction is zero, then the total angular momentum in that direction is conserved.
6. Internal torque has no influence to the total angular momentum of the system, but it will influence the angular momentum of each individual particle in the system.
7. In a non-inertial frame, the contribution of inertial torque should be considered.

Angular momentum in center of mass reference frame (Optional)

Angular momentum in center of mass reference frame (Optional)

For a point P_i, with position $\vec{r}_i$,mass m_i and velocity $\vec{v}_i$ in an arbitrary inertial reference frame. The total angular momentum of the system can be calculated by

$$\vec{L} = \sum_{i=1}^{n} \vec{L}_i = \sum_{i=1}^{n} \vec{r}_i \times m_i \vec{v}_i$$



If we consider it in the center of mass reference frame:

$$\vec{r}_i = \vec{r}_{cm} + \vec{r}_i^{\,c} \qquad \vec{v}_i = \vec{v}_{cm} + \vec{v}_i^{\,c}$$

$$\Rightarrow \vec{L} = \sum_{i=1}^{n} (\vec{r}_{cm} + \vec{r}_i^{\,c}) \times m_i (\vec{v}_{cm} + \vec{v}_i^{\,c})$$

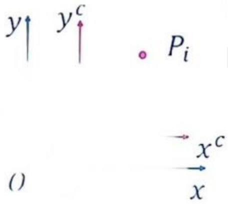
$$= \vec{r}_{cm} \times M\vec{v}_{cm} + \boxed{\sum_{i=1}^{n} \vec{r}_i^{\,c} \times m_i \vec{v}_{cm}} + \boxed{\sum_{i=1}^{n} \vec{r}_{cm} \times m_i \vec{v}_i^{\,c}} + \sum_{i=1}^{n} \vec{r}_i^{\,c} \times m_i \vec{v}_i^{\,c}$$

Equals to zero

Recap: $\vec{r}_{cm}^{\,c} = \dfrac{\sum_{i=1}^{n} m_i \vec{r}_i^{\,c}}{\sum_{i=1}^{n} m_i} = 0$, $\vec{v}_{cm}^{\,c} = \dfrac{\sum_{i=1}^{n} m_i \vec{v}_i^{\,c}}{\sum_{i=1}^{n} m_i} = 0$

Angular momentum in center of mass reference frame (Optional)

$$\vec{L} = \boxed{\vec{r}_{cm} \times M\vec{v}_{cm}} + \boxed{\sum_{i=1}^{n} \vec{r}_i^{\,c} \times m_i \vec{v}_i^{\,c}}$$



Angular momentum of CM about the reference point

Angular momentum of other parts about the CM.

The angular momentum of a system of particles about a certain point is the sum of the angular momentum associated to the movement of the center of mass about the reference point and the angular momentum associated to the movement of the particles relative to the center of mass.

Net torque caused by inertial force in CM frame (Optional)

Assume the acceleration of CM is $\vec{a}_{cm}$

Then the inertial force on i-th particle in the system $\vec{F}_i = -m_i \vec{a}_{cm}$

The torque caused by the force is

$$\vec{\tau}_i = \vec{r}_i^{\,c} \times \vec{F}_i = -\vec{r}_i^{\,c} \times m_i \vec{a}_{cm} = -m_i \vec{r}_i^{\,c} \times \vec{a}_{cm}$$

The net torque is:

$$\sum \vec{\tau}_i = \left(\sum -m_i \vec{r}_i^{\,c}\right) \times \vec{a}_{cm} \quad \Rightarrow \quad \sum \vec{\tau}_i = 0$$

Recap:

$$\vec{r}_{cm}^{\,c} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i^{\,c}}{\sum_{i=1}^{n} m_i} = 0$$

Center of mass reference frame (Optional)

In general, the motion of a particle system can be decomposed to the motion of CM and the motion of other parts about the CM.

The motion of CM is determined by: $M\vec{a}_{cm} = \vec{F}_{net-ext}$

The motion of other parts about the CM are determined by:

$$\vec{p}^{\,c}_{sys} = 0 \qquad \frac{d\vec{L}^{c}_{sys}}{dt} = \vec{\tau}_{net-ext-cm}$$

Even the CM frame is non-inertial, the net work done by the inertial force and the net inertial torque is all zero.

→ We can use the work-energy theorem and theorem of angular momentum in CM frame directly.

Theorem of angular momentum for fixed axis rotation situation

Recap: in the z direction:

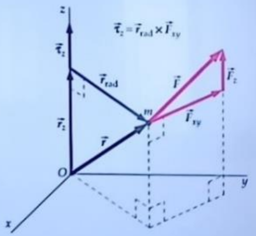
$$\vec{\tau}_z = \vec{r}_{rad} \times \vec{F}_{xy}$$

In the same manner:

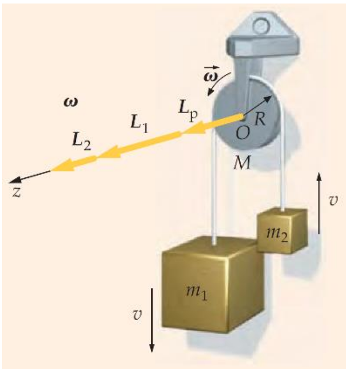
$$\vec{L}_z = \vec{r}_{rad} \times \vec{p}_{xy}$$

z component of theorem of angular momentum:

$$\vec{\tau}_{net-ext\,z} = \frac{d\vec{L}_{sys\,z}}{dt}$$



Example: An Atwood's machine has two blocks with masses m_1 and m_1 $(m_1 > m_2)$ connected by a string of negligible mass that passes over a pulley with frictionless bearings. The pulley is a uniform disk of mass M and radius R. The string does not slip on the pulley. Apply theorem of angular momentum to the system consisting of the two blocks, the string, and the pulley, to find the angular acceleration of the pulley and the linear acceleration of the blocks.



1. Let the system be everything that moves. Draw a free-body diagram of the system (Figure 10-20). The only thing touching the system is the pulley bearings. The external forces on the system are the normal force of the pulley bearings on the pulley and the gravity forces on the two blocks and the pulley:
2. Express Newton's second law for rotation, z components only (Equation 10-16):
$$\sum \tau_{\text{ext}\,z} = \frac{dL_z}{dt}$$
3. The total external torque about the z axis is the sum of the torques exerted by the external forces. The moment arms for F_{g1} and F_{g2} each equal R. (The moment arms of F_n and F_{gp} each equal zero.) $F_{g1} = m_1 g$ and $F_{g2} = m_2 g$:
$$\sum \tau_{i\,\text{ext}\,z} = \tau_n + \tau_{gp} + \tau_{g1} + \tau_{g2} = 0 + 0 + m_1 gR - m_2 gR$$
4. The total angular momentum about the z axis equals the angular momentum of the pulley, $\vec{L}_p$, plus the angular momenta of block 1, $\vec{L}_1$, and block 2, $\vec{L}_2$, each in the positive z direction. The pulley has spin angular momentum, but no orbital angular momentum because its center of mass is at rest. Each block has orbital angular momentum, but no spin angular momentum.
$$L_z = L_1 + L_2 + L_p = m_1 vR + m_2 vR + I\omega$$

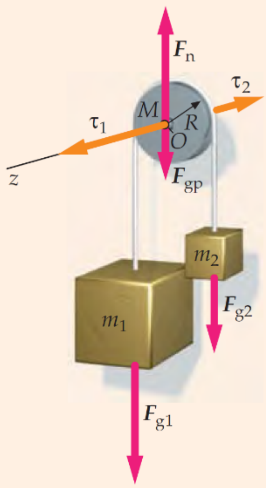


FIGURE 10-20

5. Substitute these results into Newton's second law for rotation in step 2:

$$\sum \tau_{\text{ext}\,z} = \frac{dL_z}{dt}$$

$$m_1gR - m_2gR = \frac{d}{dt}(m_1vR + m_2vR + I\omega)$$

$$m_1gR - m_2gR = (m_1 + m_2)Ra + I\alpha$$

6. Relate I to M and R, and use the nonslip condition to relate α to a and solve for both a and α:

$$m_1gR - m_2gR = (m_1 + m_2)Ra + \tfrac{1}{2}MR^2\frac{a}{R}$$

so $$a = \boxed{\frac{m_1 - m_2}{m_1 + m_2 + \frac{1}{2}M}g}$$

and $$\alpha = \frac{a}{R} = \boxed{\frac{m_1 - m_2}{m_1 + m_2 + \frac{1}{2}M}\frac{g}{R}}$$

Conservation of Angular Momentum

$$\vec{\tau}_{net} = \frac{d\vec{L}_{sys}}{dt} = 0 \Rightarrow \vec{L}_{sys} = constant$$

When the net external torque acting on a system about point remains zero, the system's angular momentum keeps constant.

General motion of a rigid body:

The degrees of freedom for a rigid body is **6**.

- Three translational degrees of freedom.
- Three rotational degrees of freedom.

The motion of the rigid body can be decomposed into translational motion of CM and rotation about the CM.

We need 6 dynamic equations to confirm the motion of the rigid body.

$$\vec{F}_{net-ext} = \frac{d\vec{p}_{sys}}{dt} \quad \text{Translational}$$

$$\vec{\tau}_{net-ext-cm} = \frac{d\vec{L}_{sys}}{dt} \quad \text{Rotational}$$

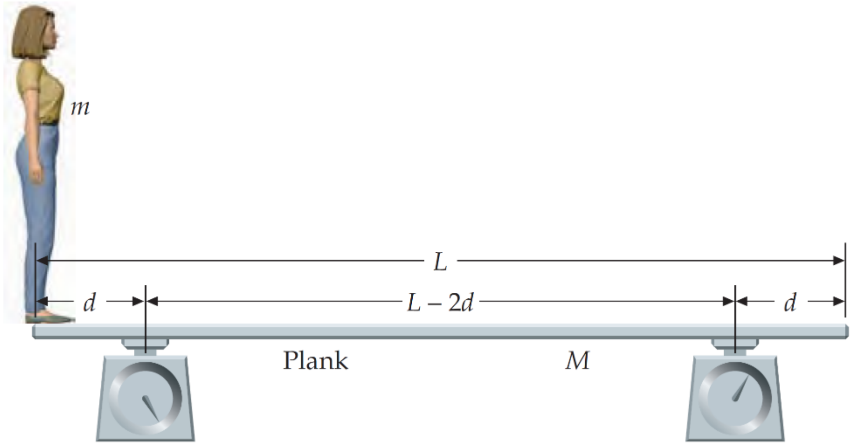
Conditions for Equilibrium:

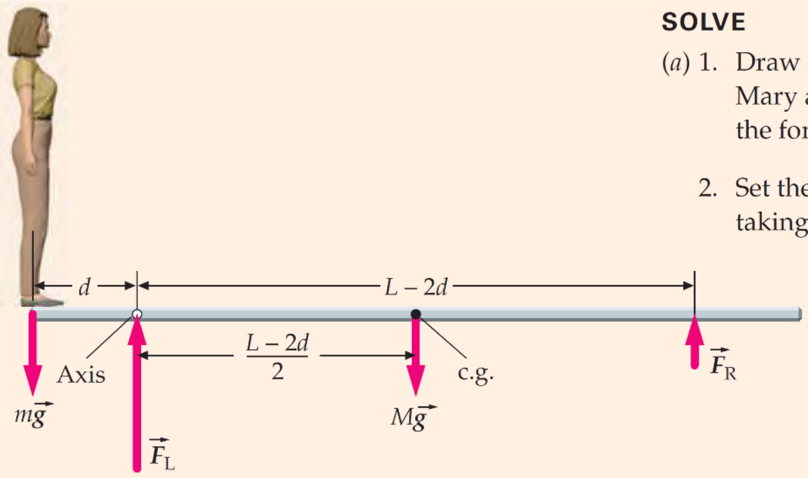
If the body stay at rest:

- The net external force acting on the body must remain zero.
- The net external torque about **any** point must remain zero.

$$\boxed{\begin{aligned}\vec{F}_{net-ext} &= 0\\ \vec{\tau}_{net-ext} &= 0\end{aligned}}$$

Example: A uniform plank of length L=3.00m and mass M=35kg is supported by scales a distance d=0.50m from the ends of the board, as shown in the right figure. (a) Find the reading on the scales when Mary, whose mass m=45kg, stands on the left end of the plank. (b) Sergio climbs onto the plank and walks toward Mary, who jumps to the floor when the plank starts to tip. Sergio keeps walking all the way to the left end of the plank, and when he gets there the scale supporting the right end of the plank reads zero. Find Sergio's mass.





SOLVE

(*a*) 1. Draw a free-body diagram of the system consisting of Mary and the plank (Figure 12-3). Forces $\vec{F}_L$ and $\vec{F}_R$ are the forces exerted by the left and right scales.

2. Set the net force equal to zero, taking upward as positive:

$$\Sigma F_y = 0$$
$$F_L + F_R - Mg - mg = 0$$

3. Calculate the net torque about the axis directed out of the page (making counterclockwise positive) and through the point of application of $\vec{F}_L$:

$$\Sigma\tau = F_L(0) + F_R(L-2d) - Mg\frac{L-2d}{2} + mgd$$

4. Set the net torque equal to zero and solve for F_R:

$$0 = F_R(L-2d) - Mg\frac{L-2d}{2} + mgd$$

$$\text{so} \quad F_R = \left(\frac{1}{2}M - \frac{d}{L-2d}m\right)g$$

5. Substitute this result for F_R into the step-2 result and solve for F_L:

$$F_L = Mg + mg - \left(\frac{1}{2}M - \frac{d}{L-2d}m\right)g = \left(\frac{1}{2}M + \frac{L-d}{L-2d}m\right)g$$

6. Substitute numerical values to obtain numerical values for the forces:

$$F_R = \left(\frac{1}{2}35\text{ kg} - \frac{0.50\text{ m}}{1.5\text{ m}}45\text{ kg}\right)(9.81\text{N/kg})$$
$$= 61.3\text{ N} = \boxed{61\text{ N}}$$

$$F_L = \left(\frac{1}{2}35\text{ kg} + \frac{2.5\text{ m}}{2.0\text{ m}}45\text{kg}\right)(9.81\text{ N/kg})$$
$$= 723\text{ N} = \boxed{7.2\times10^2\text{ N}}$$

(*b*) Using the Part-(*a*) step-4 result, set $F_R = 0$ and solve for m:

$$0 = \left(\frac{1}{2}M - \frac{d}{L-2d}m\right)g$$

$$\text{so} \quad m = \frac{L-2d}{2d}M = \frac{2.0\text{ m}}{1.0\text{ m}}35\text{ kg} = \boxed{70\text{ kg}}$$

Equilibrium 平衡

Conditions for Equilibrium:

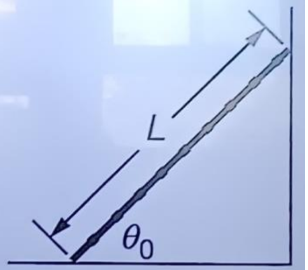
A uniform ladder of mass M and length L rests against a smooth wall at an angle θ_0, as shown in the figure.

a. What is the torque due to the weight of the ladder about its base?
b. Determine the normal force of the wall act on the ladder.
c. Determine the frictional force between the ladder and the ground.

a. $\tau = \dfrac{MgL\cos\theta_0}{2}$

b. $F_N = \dfrac{Mg}{2\tan\theta_0}$

c. $F_f = \dfrac{Mg}{2\tan\theta_0}$



Static Equilibrium in an accelerated frame:

(Optional)

A truck carries a uniform block of marble of mass m, height h, and square cross section of edge-length L. What is the greatest acceleration the truck can have without the block tipping over? Assume that the block tips before it slides.

Summary 总结

1. Velocity and Acceleration

Velocity $v = dx/dt$	Angular velocity $\omega = d\theta/dt$
Acceleration $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$	Angular acceleration $\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$

Tangential velocity $v_t = r\omega$

Tangential acceleration $a_t = r\alpha$

Centripetal acceleration $a_n = v^2/r = \omega^2 r$

2. Constant acceleration

$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$x = x_0 + vt + \frac{1}{2}at^2$	$\theta = \theta_0 + \omega t + \frac{1}{2}\alpha t^2$
$v^2 = v_0^2 + 2a(x - x_0)$	$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$

3. Moment of Inertia

Systems of particles $I = \sum m_i r_i^2$

Continuous object $I = \int r^2 dm$

Parallel-axis theorem $I = I_{cm} + Md^2$

4. Torque

$\tau = F_t r = Fr\sin\varphi = Fl$

5. Newton's Second law

Particle	Plane-parallel motion
$\sum \vec{F} = m\vec{a}$	$\begin{cases} F_{ext-x} = ma_{cm-x} \\ F_{ext-y} = ma_{cm-y} \\ \tau_{ext-cm} = I_{cm}\alpha \end{cases}$

6. Non-slip Condition

Rolling without slipping

$v_{cm} = r\omega$

$a_{cm} = r\alpha$

Point P with position $\vec{r}$ on the rigid body

$\vec{v}_p = \vec{v}_{cm} + \vec{\omega} \times \vec{r}$

If the velocity of the contact point A is zero

$v_{cm} = \omega r$

7. Work and energy

Kinetic energy $E_k = \frac{1}{2}mv^2$	Kinetic energy $E_k = \frac{1}{2}I\omega^2$
Power $P = FV$	Power $P = \tau\omega$
	Kinetic energy for rotating object $E_k = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2$

Work-energy theorem for rigid body:

$$\Delta E_k = E_k(t) - E_k(0) = \int_0^t \vec{F}_{ext} \cdot d\vec{r} + \int_{\theta_0}^{\theta} \tau_{ext-cm}\, d\theta$$

F_{ext} on its center of mass

τ_{ext} on its center of mass

8. Angular Momentum	
Momentum $\vec{p}=m\vec{v}$	Angular Momentum $\vec{L}=\vec{r}\times\vec{p} \qquad (kg\cdot m^2/s)$
Theorem of angular momentum: $\vec{\tau}_{net}=\frac{d\vec{L}}{dt}$ Integral representation: $\Delta\vec{L}=\int_0^t \vec{\tau}_{net}dt$ Angular impulse: $\vec{J}=\int_0^t \vec{\tau}_{net}dt \Rightarrow d\vec{L}=d\vec{J}$	Net torque act on the object equals to the change rate of angular momentum of the object about the same point. The change of angular momentum of the object equals to the angular impulse exert on the object about the same point.
Conservation of Angular Momentum $\vec{\tau}_{net}=\frac{d\vec{L}_{sys}}{dt}=0 \Rightarrow \vec{L}_{sys}=constant$	

目录